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Critical dynamics of the $d = 1$ kinetic Ising model

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Abstract. The critical dynamics of the $d = 1$ alternating bond Glauber–Ising model and related models is studied. There are two contributions to the critical slowing down. One is due to the long-range fluctuations near the critical point. It is characterized by the dynamic exponent $z = 2$. The other contribution is a temperature dependence of the bare time scale, which is a result of short-range phenomena. In contrast to previous studies, it is shown that all these models belong to one universality class.

1. Introduction

The theory of dynamic scaling [1,2] relates the critical slowing down near the critical temperature T_c to the long-range fluctuations of the order parameter. The generalization of the static scaling hypothesis to time-dependent phenomena was suggested before the development of the renormalization group (RG) technique [3]. This generalization is straightforward: near T_c there is one relevant time scale, τ_c. This time scale depends on $T - T_c$ only through its dependence on the relevant length scale of the system, the correlation length $\xi(T - T_c)$, and thus may diverge at T_c with a characteristic exponent z.

The static scaling hypothesis leads to the classification of the critical exponents in terms of universality classes and it shows that only a few parameters of a system are important for the nature of the fixed points of the RG transformation. These parameters include the symmetry of the Hamiltonian and the dimensionality of the system. Since these parameters are common to a variety of systems, it is possible to classify the behaviour of systems near T_c into a few universality classes.

The main theoretical support for the dynamic scaling hypothesis came, as in the statics, from the RG study of dynamic phenomena. Halperin *et al* [2] showed that the dynamic exponent z, $\tau_c \sim \xi^z$, of different systems can also be classified according to universality classes. These classes are the subdivision of the static universality classes according to the conservation laws that the systems obey. The behaviour of a system at different wavevectors k near the critical temperature is characterized according to three regions

$$\begin{aligned} &\text{region I}: k\xi \ll 1 \qquad T - T_c < 0 \\ &\text{region III}: k\xi \ll 1 \qquad T - T_c > 0 \\ &\text{region II}: k\xi \gg 1 \qquad T \approx T_c. \end{aligned} \tag{1}$$

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The dynamic scaling hypothesis predicts the following asymptotic behaviour for the characteristic time scale, $\omega_c(k) \equiv \tau_c^{-1} = k^z \Omega(k\xi)$, in the different regions:

$$k\xi \to \infty \text{(region II)}: \omega_c(k) = k^z \Omega(k\xi) \qquad \Omega(k\xi) \to \text{constant}$$
$$k\xi \to 0 \text{(region III)}: \omega_c(k) = \xi^{-z} f(k\xi) \qquad f(k\xi) \to \text{constant}. \tag{2}$$

Note that region I is inaccessible if $T_c = 0$.

Following the study [2] of the critical dynamics of systems having dimension $d = 4-\epsilon$, real-space RG studies of systems having lower dimension were reported [4, 5]. Neither techniques displayed any violations of the dynamic scaling hypothesis. However, in the last half decade, several authors have predicted that dynamic scaling can be violated in systems having a zero-temperature critical point. Studies of quasi-one-dimensional systems with an underlying fractal or hierarchical structure suggested a temperature-dependent dynamic exponent [6, 7]. Haake and Thol [8] showed that z of the $d = 1$ Ising model may depend on the details of the kinetic equation. On the other hand, exact RG calculations (in the linear response regime) are consistent with the 'conventional' dynamic scaling hypothesis [9, 10]. Kutasov *et al* [7] attempted to explain this controversy using a model with hierarchical couplings. They used RG methods and a matching procedure [11, 12], and obtained a result similar to that of Henley [6]. Droz *et al* [13] pointed out that if the $d = 1$ Ising system has a alternating-bond structure, z depends on the ratio of the bond strengths. Therefore, any value of $z \geqslant 2$ can be found. Ashraff and Stinchcombe [14] performed exact RG calculations for the alternating-bond Ising chain and obtained a non-universal value of z in agreement with Droz *et al* [13]. Lage [15] has studied a model with a larger unit cell and also obtained a non-universal value for z using RG methods. However, we have re-examined these calculations and have shown that, by carefully separating the critical effects governed by the fixed point of the RG transformation from the non-critical effects due to short-range details which determine the amplitudes, a universal value of z is obtained.A similar controversy for the alternating-bond Ising chain also appears in the problem of spinodal decomposition [16, 17]. In a recent letter [18] we have shown that for a chain with alternating-bond strengths there exists a fundamental difference between the single- and multi-spin flip dynamics. The reason for the apparent violation of standard scaling is that the critical point of the system is at zero temperature. Any inhomogeneous Ising chain has a large number of states at zero temperature which are metastable [19] against single-spin flips. Spins that are coupled by strong bonds form blocks which have their spins aligned. At low T, the lowest energy excitations correspond to these blocks flipping as a unit because in this case only the weak bonds connecting the blocks are broken. Hence the important excitations in the model do not correspond to single-spin flips. Therefore, if only single-spin flip dynamics is allowed, there are an infinite number of divergent relaxation times with each one corresponding to a metastable state. By allowing two-spin flip dynamics, we demonstrated that the standard dynamics is recovered and that the alternating-bond chain belongs to the same dynamic universality class as the homogenous Glauber Ising chain. It is straightforward to generalize this treatment to more complicated structures with a larger unit cell.

In the present work we study the discrepancy between the results obtained by RG and by other methods in one-dimensional systems. We address the question of the validity of conventional dynamic scaling in these systems when the system is restricted to single-spin flips. A common feature of these systems is that at the critical point they are also in the ground state. Therefore, near the zero-temperature point, the variation in the bond strengths can give rise to metastable states consisting of finite blocks that have a diverging bare time

scale. By 'bare time scale' we mean that its origin is due to short-range effects. The bare time scale has nothing to do with the long-range fluctuations that characterize the critical point. These metastable states do not affect the static critical exponents but they are directly responsible for the presence of non-universal temperature-dependent terms in the dynamics. The reported violations of the dynamic scaling hypothesis are in region III where $k \to 0$ and it is difficult to separate the 'bare time scale' from the critical contribution. The inability to distinguish in region III between the two kinds of contributions caused some authors [8, 13, 17, 20] to interpret the results of their study as a violation of dynamic scaling.

The outline of the present paper is as follows. A system with alternating-bond strengths is described in section 2 and it is shown that the conventional dynamic scaling result of $z = 2$ is correct and that the apparent violations are due to a temperature dependence of the bare time scale. In section 3, it is shown that the above result is a common feature to all $d = 1$ kinetic Ising models that are translationally invariant. All of these systems do belong to one universality class with the dynamic critical exponent $z = 2$. The application of the standard RG techniques to these systems is reviewed in section 4. It is shown that, although the fixed points of these methods are not sensitive to the short-range details, a careful analysis of the approach to the unstable fixed points must be carried out in order to calculate the effects on the bare time scale. Our results are summarized in section 5.

2. The kinetic Ising model with alternating-bond strengths

The kinetic Ising model on the linear chain with alternating bonds is the simplest non-trivial model that presents the controversial phenomena discussed in the previous section. It has the advantage that it can be solved exactly by a variety of methods. The Hamiltonian (in units of $-1/k_B T$) of the Ising chain with alternating bonds has the form

$$H = \sum_i J_i S_{i-1} S_i \qquad J_{2i} = J_1 \qquad J_{2i+1} = J_2 \tag{3}$$

where $J_2 > J_1 \geqslant 0$. The equilibrium properties of the partition function, $P_e = e^H$, were studied using RG methods by Nelson and Fisher [12]. These properties can also be evaluated directly using transfer matrix methods.

The master equation of the Glauber [21] kinetic Ising model is

$$\frac{d}{dt} P(\{S\}, t) = -\Gamma \sum_i (1 - p_i) W_i(S_i) P(\{S\}, t) \tag{4}$$

where $P(\{S\}, t)$ is the time-dependent spin probability distribution, Γ^{-1} is a bare time scale for single-spin flips, p_i is the spin flip operator, $p_i f(S_i) = f(-S_i)$, and $W_i(S_i)$ is the probability transition rate of the ith spin from a state S_i to a state $-S_i$.

The transition probability rate satisfies detailed balance,

$$(1 - p_i) W_i(S_i) P_e(\{S\}) = 0 \tag{5}$$

and thus ensures the ergodicity of the system [21]. However, this equation does not define a unique W_i. The straightforward generalization of the choice made by Glauber to the alternating bond chain [13] is

$$W_i^G = \tfrac{1}{2}(1 - a_i^+ S_{i-1} S_i - a_i^- S_i S_{i+1}) \tag{6}$$

where

$$a_i^{\pm} \equiv \tfrac{1}{2}[\tanh(J_i + J_{i+1}) \pm \tanh(J_i - J_{i+1})]. \tag{7}$$

The average spin value is the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} q_l \equiv \frac{\mathrm{d}}{\mathrm{d}t}\langle S_i \rangle = -2\Gamma\langle S_i W_i(S_i)\rangle \tag{8}$$

where the average $\langle\ \rangle$ is taken with respect to $P(\{S\}, t)$. Using (7), the set of equations (8) is reduced in the ith unit cell to the following equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} q_{2i} &= -\Gamma(q_{2i} - a_1^{+} q_{2i-1} - a_1^{-} q_{2i+1}) \\ \frac{\mathrm{d}}{\mathrm{d}t} q_{2i+1} &= -\Gamma(q_{2i+1} - a_2^{+} q_{2i} - a_2^{-} q_{2i+2}) \end{aligned} \tag{9}$$

where

$$a_2^{\pm} = a_1^{\mp} \equiv \tfrac{1}{2}(\tanh(J_2 + J_1) \pm \tanh(J_2 - J_1)). \tag{10}$$

A summation with respect to i of the set of equations (9) leads to a simple solution for the magnetization $M(t) = \sum_i q_i$. Droz *et al* [13] found that the leading term in the magnetization at low T is

$$M(t) = M(0)\exp\{-\Gamma[1 - \tanh(J_2 + J_1)]t\}. \tag{11}$$

Thus it appears that the magnetization relaxes near the critical point $T_c = 0$ with a time scale $\tau \sim (2\Gamma)^{-1}\mathrm{e}^{2(J_2+J_1)}$. The correlation length, ξ, is controlled by the weakest bond [12] and diverges as $\xi \sim \mathrm{e}^{2J_1}$. By expressing τ in terms of ξ, Droz *et al* concluded that $\tau \sim \xi^{1+J_2/J_1}$ and hence that the system has a non-universal dynamic exponent $z = 1 + J_2/J_1$.

The equations in (9) can be solved for any Fourier component of $M(t)$. This process reveals that, while the first part of their conclusion is formally correct, the second part can be re-interpreted. The dependence of the time scale on the wavevector can be calculated by substituting $q_{2l+1} = Q_1 \exp\{\mathrm{i}[(2l+1)k - \omega t]\}$ and $q_{2l} = Q_2 \exp\{\mathrm{i}[2lk - \omega t]\}$ into (9). The secular equation leads to the following dispersion relation

$$\omega/\Gamma = 1 \pm [\tanh^2(J_2 + J_1) + (\tanh^2(J_2 - J_1) - \tanh^2(J_2 + J_1))\sin^2 k]^{1/2}. \tag{12}$$

Luscombe [20] also obtained this result but did not identify the value of z correctly. In the limit of small k, the slow mode behaves as $\omega_c \sim \Gamma\mathrm{e}^{-2(J_2-J_1)}(\xi^{-2} + k^2)$. This can be written as

$$\omega_{\mathrm{II}} \sim \Gamma\mathrm{e}^{-2(J_2-J_1)}k^2(1 + (k\xi)^{-2})$$

in region II and as

$$\omega_{\mathrm{III}} \sim \Gamma\mathrm{e}^{-2(J_2-J_1)}\xi^{-2}(1 + (k\xi)^2)$$

in region III. Comparison with (2) identifies the universal value $z = 2$ of the standard kinetic Ising model [21, 9] but with a temperature-dependent bare time scale $\Gamma^{-1}(T)$.

Thus the model has two time scales. The first one is the bare time scale

$$\tau_0 \sim \Gamma^{-1} e^{2(J_2 - J_1)} \tag{13}$$

which is due to short-range effects and characterizes the width of the spectrum. The dispersion relation (12) describes two bands of characteristic frequencies and is shown in figure 1. In the limit of low T the width of both bands is proportional to $e^{-2(J_2-J_1)}$. Each mode in the lower band corresponds to a single-spin-flip metastable state with a relaxation time that diverges at $T = 0$. However, this divergence has nothing to do with critical phenomena and is present in any chain with inhomogeneous couplings. When all the bonds have the same strength, the two bands merge into each other and the width remains finite at $T = 0$. If the characteristic frequencies are measured in units of the band width, then the internal structure of the dispersion relation is the same as in the homogeneous chain. The second time scale (in units of the first), $\sim \xi^2$ is due to the long-range fluctuations. This time scale is responsible for the detailed structure of the dispersion relation. Therefore, when the bare time scale due to short-range effects is separated from the time scale due to long-range fluctuations, the dynamic exponent z is clearly identified and has the same value as in the homogeneous chain.

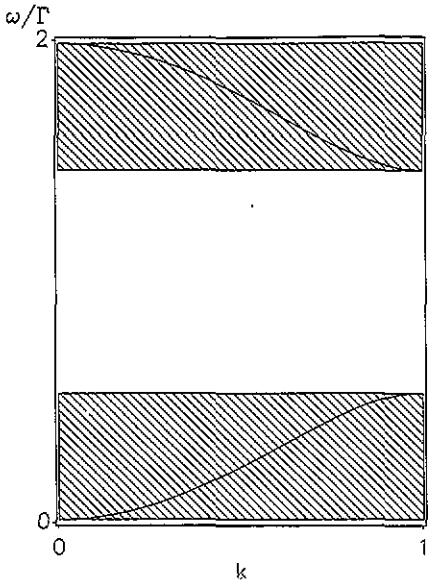

Figure 1. The characteristic frequencies ω/Γ as a function of k (in units of $\pi/2$) for the alternating-bond chain with $J_2 = 1.5, J_1 = 1.0$ form two bands as indicated by the shaded areas.

3. Critical dynamics of Ising chains with large unit cells

The dynamics of the alternating-bond Glauber–Ising chain is the conventional Glauber dynamics [21] combined with a diverging bare time scale associated with short-range single-spin-slip metastable states separated by weak bonds. The same dynamics is expected even when the bond structure is more complicated provided it is translationally invariant.

The critical dynamics, as well as the statics are controlled by the weakest bonds, $J_{\min}$, and this ensures the Glauber dynamics with $z = 2$. The metastable states that have diverging bare time scales are configurations in which the spins in the cells between the $J_{\min}$ are aligned

in the same direction. The bare time scale is associated with the maximum energy barrier of the metastable state, $\tau_0 \sim e^{2(J_{max}-J_{min})}$, where J_{max} is the maximum bond strength.

To prove this result one has to find the asymptotic form of the acoustic branch of the secular equation given by the determinant of an $n \times n$ matrix, where $n > 2$ is the number of spins in the unit cell of the Ising chain. The main steps in the long, but straightforward, algebra are reported in the appendix. The interactions in the unit cell can vary without any general pattern. In the following we present only the model and the results.

The Hamiltonian (in units of $-1/k_BT$) of the general Ising chain is

$$H = \sum_i J_i S_{i-1} S_i \qquad J_i = J_{i+n} \tag{14}$$

and all J_i are positive. The probability transition rate (6) is also applied to the general case. The set of equations (8) is reduced to n equations in the unit cell

$$\frac{d}{dt} q_j = -\Gamma(q_j - a_j^+ q_{j-1} - a_j^- q_{j+1}) \qquad j = 1, n \tag{15}$$

where a_j^+, a_j^- are given by (7). If we assume a solution of the form, $q_j = Q_j e^{i[jk-\omega t]}$, we obtain the secular equation, $|\mathbf{D}| = 0$ where $\mathbf{D}$ has the form,

$$\mathbf{D} = \begin{bmatrix} (1-\omega/\Gamma) & -a_1^- e^{ik} & 0 & 0 & \dots & -a_1^+ e^{-ik} \\ -a_2^+ e^{-ik} & (1-\omega/\Gamma) & -a_2^- e^{ik} & 0 & \dots & 0 \\ 0 & -a_3^+ e^{-ik} & (1-\omega/\Gamma) & -a_3^- e^{ik} & \dots & 0 \\ \vdots & & & & & \\ -a_n^- e^{ik} & 0 & \dots & 0 & -a_n^+ e^{-ik} & (1-\omega/\Gamma) \end{bmatrix}. \tag{16}$$

This matrix is an almost tridiagonal matrix (the 'almost' is due to the elements in the upper right and lower left corners). The form of the determinant is typical of tight binding problems and the rules for its expansion are described in the appendix. At low T the asymptotic form of the elements in $\mathbf{D}$ is

$$\begin{aligned} &a_l^+ \sim 1 - e^{-2\Delta_l} \qquad a_l^- \sim e^{-2\Delta_l} \qquad \text{if } J_{l+1} - J_l < 0 \text{ (decreasing bonds)} \\ &a_l^+ \sim e^{-2\Delta_l} \qquad a_l^- \sim 1 - e^{-2\Delta_l} \qquad \text{if } J_{l+1} - J_l > 0 \text{ (increasing bonds)} \end{aligned} \tag{17}$$

where $\Delta_l = |J_{l+1} - J_l|$. Using these low-temperature forms we find the following dispersion relation for the slowest mode (see the appendix)

$$\omega/\Gamma \sim e^{-2(J_{max}-J_{min})}[1 - \cos(nk) + e^{-4J_{min}}]. \tag{18}$$

Again we have two different time scales. The factor $\tau_0 \sim \Gamma^{-1} e^{2(J_{max}-J_{min})}$ is the local unit cell time scale and is not related to the critical fluctuations. The critical dynamics is similar to that of the alternating-bond model and can be written in regions II and III respectively as

$$\begin{aligned} \omega_{II} &\sim \Gamma e^{-2(J_{max}-J_{min})} k^2 [1 + (nk\xi)^{-2}] \\ \omega_{III} &\sim \Gamma e^{-2(J_{max}-J_{min})} \xi^{-2} [1 + (nk\xi)^2]. \end{aligned} \tag{19}$$

The correlation length, $\xi \sim e^{2J_{min}}$, is a function of the weakest bond in the unit cell [12]. Comparison of (19) with (2) identifies the universal value $z = 2$ of the standard kinetic Ising model.

RG studies [14, 15] of the models in this and the previous section have found non-universal values for z which depend upon the magnitides of the bonds. However, in the following sections we show that a careful analysis of these RG methods does yield a universal value for z for all of these models.

4. RG study of the kinetic Ising model

4.1. Decimation with scaling factor b = 2

The study of the alternating-bond Ising chain clearly shows that there can be different contributions to the time scale of the system which are difficult to separate in region III. Since most of the RG calculations are performed in this region, it is of interest to study the alternating-bond model in the RG context. There are many schemes of RG transformations that can be used to study the master equation (4). The simplest approach is the decimation of the master equation (8) for the average spin value [22].

In the set of equations (9) that describes the kinetics of the alternating-bond chain, we can substitute q_{i-1} and q_{i+1} into the equation for q_i. This eliminates half of the degrees of freedom, and creates an effective spin system on a lattice with double the lattice spacing (an RG transformation with scaling factor $b = 2$). The Laplace transform of the spin average Q_{2i}, satisfies in the limit $\omega \to 0$ the following equation

$$\left(1 - \frac{2\omega}{\Gamma(1 - a_1^+ a_2^- - a_2^+ a_1^-)}\right) Q_{2i} = \frac{a_1^+ a_2^+}{1 - a_1^+ a_2^- - a_2^+ a_1^-}(Q_{2i-2} + Q_{2i+2}). \tag{20}$$

Using the definitions (10) for $a_i^{\pm}$ it is easy to see that (20) is equivalent to the master equation of the uniform chain with an effective interaction $\tilde{J}$ given by

$$\tanh \tilde{J} = (\tanh J_1)(\tanh J_2) \tag{21}$$

and an effective bare inverse time scale

$$\tilde{\Gamma} = \Gamma e^{-2(J_2 - J_1)}. \tag{22}$$

In terms of the renormalized quantities after the first step, equation (20) has the form

$$\left(1 - \frac{\omega}{\tilde{\Gamma}}\right) q_i = \tfrac{1}{2} \tanh(2\tilde{J})(q_{i-1} + q_{i+1}). \tag{23}$$

The effective uniform chain can now be renormalized again but the effective bare time scale remains unchanged since the bonds are now homogeneous on this length scale. The static RG transformation is $\tanh \tilde{J}' = \tanh^2 \tilde{J}$ and the critical fixed point of the RG transformation is $\tanh \tilde{J}^* = 1$ which yields

$$\tanh J_1^* = \tanh J_2^* = 1. \tag{24}$$

The RG transformation of the time scale of the uniform Ising chain can be obtained as the limit $J_1 = J_2$ of the factor $2/(1 - a_1^+ a_2^- - a_2^+ a_1^-)$ in (20) and is given by

$$\tau' = \tfrac{1}{2}(1 - \tanh^2 \tilde{J})\tau. \tag{25}$$

The rescaling of the time (25) at the fixed point (24) leads to the dynamic exponent

$$\tau' = b^{-z}\tau \qquad b^z = 4 \qquad z = 2. \tag{26}$$

The above RG scheme gives the correct z that characterizes the slowing down due to the long-range fluctuations near T_c. The characteristic time scale is, however, measured in units of the diverging bare time scale $\tilde{\Gamma}^{-1}$. After a Fourier transform of (23) in k space, the dispersion relation in region III becomes

$$\begin{aligned} \omega &\sim 2\tilde{\Gamma} e^{-4\tilde{J}}[1 + \tfrac{1}{2}k^2] \\ &\sim 2\Gamma e^{-2(J_2 - J_1)} e^{-4J_1}[1 + e^{4J_1} k^2]. \end{aligned} \tag{27}$$

The two contributions to the critical slowing down are clearly separated.

4.2. Decimation with scaling factor $b = 3$

The decimation by a scaling factor $b = 2$ does not preserve the alternating-bond nature of the original problem. When the decimation is performed with an odd scale factor, the bond structure is preserved [12]. The set of equations (9) can be written using reduced variables

$$x_i^{\pm} = \frac{a_i^{\pm}}{1 - \omega/\Gamma} \tag{28}$$

as

$$Q_{2i} = x_1^{+} Q_{2i-1} + x_1^{-} Q_{2i+1} \qquad Q_{2i+1} = x_2^{+} Q_{2i} + x_2^{-} Q_{2i+2}. \tag{29}$$

The Q_{2i+j} with $j = \{\pm 2, \pm 1\}$ are eliminated from this set of equations. The remaining $Q's$ satisfy equations of the same form with renormalized coefficients corresponding to the scaling factor $b = 3$

$$Q_{2i} = x_1^{+'} Q_{2i-3} + x_1^{-'} Q_{2i+3}. \tag{30}$$

where

$$x_1^{+'} = \frac{x_1^{+2} x_1^{-} (1 - x_1^{+2})}{(1 - x_1^{+2} + x_1^{+} x_1^{-} - x_1^{-2})(1 - x_1^{+2} - x_1^{+} x_1^{-} - x_1^{-2})} \tag{31}$$

and $x_1^{-'}$ is obtained by interchanging the plus and the minus signs. The relationships between the parameters $x_1^{\pm'} = x_2^{\mp'}$ are preserved. The above transformation is the same as that obtained by Ashraff and Stinchcombe [14].

The non-trivial fixed point of (31) is

$$x_i^{+*} = x_i^{-*} = \tfrac{1}{2}. \tag{32}$$

This implies that $\omega^* = 0$ and $T_c = 0$. At $\omega = 0$, the recursion relations are equivalent to the known static recursion relations [12]

$$\tanh J_1' = (\tanh^2 J_1)(\tanh J_2) \qquad \tanh J_2' = (\tanh J_1)(\tanh^2 J_2) \tag{33}$$

which have the fixed point

$$\tanh J_1^* = \tanh J_2^* = 1. \tag{34}$$

The trajectory flow of the RG transformation (31) is shown in figure 2. There are two fixed points: the high-temperature fixed point O which is attractive and the homogeneous chain critical point A (32) which has one repulsive and one attractive eigenvalue. The eigenvalues of the linearized recursion relations at A are 9 and $\frac{1}{3}$ and the corresponding eigenvectors are in the direction of the two invariant lines $x_1^{+} = x_1^{-}$ and $x_1^{+} + x_1^{-} = 1$ respectively. At low T, the initial state of the alternating-bond Hamiltonian is located in the vicinity of point B. The flow from the initial state to the region near A cannot be described by an RG transformation linearized about the point A. To describe the flow along the line BA it is suffucient to keep only terms up to order $\epsilon = e^{-2(J_2 - J_1)}$ and we have

$x_1^+ = 1 - \epsilon$, $x_1^- = \epsilon$. In the low-frequency and small k limit, the dispersion relation (12) can be written in terms of the reduced variables as

$$\omega/\Gamma \sim [1 - (x_1^+ + x_1^-)] + \frac{1}{2}\left[\frac{(x_1^+ + x_1^-)^2 - (x_1^+ - x_1^-)^2}{(x_1^+ + x_1^-)^2}\right]k^2. \tag{35}$$

Near B, we have $x_1^+ + x_1^- \sim 1$ and $x_1^+ - x_1^- \sim 1 - 2\epsilon$ and this gives

$$\omega/\Gamma \sim 2\epsilon k^2. \tag{36}$$

However, the first iteration of the transformation moves the initial point B very close to the fixed point A. In the latter case we still have $x_1^+ + x_1^- \sim 1$ but now $x_1^+ - x_1^- \sim 0$ which yields

$$\omega/\tilde{\Gamma} \sim \tfrac{1}{2}k^2. \tag{37}$$

Hence the first step of the transformation is equivalent to a renormalization of the bare inverse time scale Γ^{-1} by the factor $e^{2(J_1+J_2)}$.

To find the flow of $(x_1^+ + x_1^-)$ away from the line BA, the next term, $e^{-2(J_2+J_1)}$ must be taken into account. The eigenvalue in this repulsive direction is, in general, b^2 which gives the dynamic exponent $z = 2$.

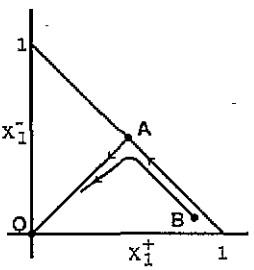

Figure 2. Schematic representation of the flow of the $b = 3$ RG transformation from the initial low-temperature point B. The critical fixed point is located at A and the high-temperature fixed point is at O.

The flow of the trajectories from the point B clearly demonstrates that the alternation of the bonds is irrelevant in contrast with the interpretation given by Ashraff and Stinchcombe [14]. The irrelevant parameter, $x_1^+ - x_1^-$, flows rapidly to zero. However, in the first RG step it multiplies the time scale by a temperature-dependent term as in (22). After this first step of the RG transformation the model becomes an effective homogeneous one.

4.3. RG transformation of the master equation

In the Glauber–Ising chain model the magnetization is an eigenfunction of the Liouville operator. Therefore it is easy to transform the master equation (4) to an equation for the average magnetization, and then decimate it as described above. A different approach [4] that suits more complicated cases is to renormalize the master equation itself. This can be done by linearizing the deviations of probability distribution function from equilibrium as follows

$$\Phi = P(\{S\}, t)/P_e(\{S\}) = 1 + h(t)\hat{O}(\{S\}) \qquad \hat{O}(\{S\}) = \sum_i S_i. \tag{38}$$

The master equation becomes

$$P_e(\{S\})\frac{d}{dt}h(t)\hat{O}(\{S\}) = -\mathcal{L}P_e(\{S\})\hat{O}(\{S\})h(t) \tag{39}$$

where

$$\mathcal{L}(\{S\}) \equiv \Gamma \sum_i (1-p_i)W_i(S_i)$$

and, for convenience [4], the transition rate is taken to be

$$W_i^A = \left[\frac{P_e(\{S\}', -S_i)}{P_e(\{S\}', S_i)}\right]^{1/2}. \tag{40}$$

This form of the transition rate is related to the usual Glauber rate (6) as follows

$$W_i^G = W_i^A e^{-(J_2-J_1)}. \tag{41}$$

The decimation of $(b-1)$ spins out of every b spins along the chain in the terms on both sides of (39) transforms the equation to a similar one but with renormalized parameters

$$h' = bh \qquad t' = \Omega t \tag{42}$$

where

$$\Omega = \frac{\cosh J_1' \cosh J_2'}{\cosh J_1 \cosh J_2} b^{-1} \tag{43}$$

and J_1', J_2' satisfy the static recursion relations (33). At low T the first iteration of the RG transformation gives $\Omega \sim e^{-(J_2-J_1)}$. For further iterations Ω is equal to b^{-2} which yields $z = 2$. Taking into account the normalization factor in (41), we find that the characteristic time scale is again of the form $\tau \sim \Gamma^{-1} e^{2(J_2-J_1)} \xi^2$.

5. Conclusions

We have shown that a large class of inhomogeneous bond Ising models belong to the same universality class as the uniform bond model. They are all characterized by the dynamic exponent $z = 2$ but the bare time scale has a non-universal temperature dependence which is absent in the uniform case. Both exact diagonalization and exact RG methods were used to separate these two contributions. This is especially important for transitions at $T_c = 0$ since the bare time scale can diverge. The fact that these two contributions were not separated in previous studies [13–15, 17, 20, 23] predicts a violation of the standard dynamic scaling in inhomogeneous Ising systems. A second problem where the separation of these two contributions is important is for spinodal decomposition in the alternating-bond Ising chain [16, 17]. According to Bray [16] the domain growth in the Glauber–Ising chain, following a sudden quench from an initial high-temperature state to a small non-zero temperature, is described by a power-law decay of the autocorrelation function

$$C(0, t, t') \sim (1/t')^{1/z} \tag{44}$$

for constant t. This dependence was confirmed in the homogenous Ising chain [17, 24]. Our results can be used to predict the decay of the autocorrelation function in the alternating-bond Glauber–Ising model. Since it belongs to the uniform bond universality class, it has $z = 2$. However, in the long-time region, the time is measured in terms of the bare time scale of the specific kinetic model. For the alternating-bond Glauber model the temperature-dependent bare time scale is $\Gamma^{-1}e^{2|J_2-J_1|}$. This is exactly the expression found in [17]. However, the unusual scaling form suggested in that reference is not needed.

Finally, our results also apply to systems with two types of bonds distributed randomly [15, 23] along the chain. It is the weakest bond that determines the asymptotic behaviour of the correlation length. The regions between these weak bonds form blocks which at $T = 0$ are metastable against single-spin flips. These metastable states form a band at finite temperature with a width proportional to $e^{-(J_2-J_1)}$. Hence the bare time scale is strongly temperature-dependent but the dynamic exponent $z = 2$ and the random bond model belongs to the same universality class as the uniform chain.

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Appendix. The secular equation

The determinant of the almost tridiagonal $n \times n$ matrix $|\mathbf{D}|$ (16) is

$$\left\{\sum_{j=0}^{j_{max}}(-1)^j\left(1-\frac{\omega}{\Gamma}\right)^{n-2j}A_j^n\right\}-e^{-ink}\prod_j a_j^+-e^{+ink}\prod_j a_j^- \tag{A1}$$

where $j_{max} = \text{Integer}[\frac{1}{2}n]$. The first term above is due to the tridiagonal part of $|\mathbf{D}|$ whereas the last two terms are due to the presence of the elements in the upper right and lower left corners respectively. These latter terms are the only k-dependent terms. The determinant of the tridiagonal part of $\mathbf{D}$ satisfies the recursion relation

$$\mathbf{D}_n = (1-\omega/\Gamma)\mathbf{D}_{n-1} - a_{n-1}^- a_n^+ \mathbf{D}_{n-2}. \tag{A2}$$

This relation is equivalent to the following recursion formula for the A_j^n

$$A_j^n = A_j^{n-1} + A_{j-1}^{n-2} a_{n-1}^- a_n^+ \qquad (A_0^n \equiv 1). \tag{A3}$$

The A_j^n are the sum of products of j factors of the form $a_{i-1}^- a_i^+$. The low T form of these factors depends on the magnitudes of the bonds (7) J_{i-1}, J_i, J_{i+1}. The leading terms have the form

$$a_{i-1}^- a_i^+ \sim \begin{cases} (1-e^{-2\Delta_{i-1}}-e^{-2\Delta_i}) & J_i > \max\{J_{i-1}, J_{i+1}\} \\ e^{-2\Delta_i}(1-e^{-2\Delta_{i-1}}) & J_i < J_{i+1}, J_i > J_{i-1} \\ e^{-2\Delta_{i-1}}(1-e^{-2\Delta_i}) & J_i < J_{i-1}, J_i > J_{i+1} \\ e^{-2\Delta_{i-1}}e^{-2\Delta_i} & J_i < \min\{J_{i-1}, J_{i+1}\} \end{cases} \tag{A4}$$

where $\Delta_i = |J_{i+1} - J_i|$. This approximation was derived by replacing the term $\tanh(J_i + J_{i+1})$ by 1. The next correction is obtained by the following replacement in each term above

$$e^{-2\Delta_i} \rightarrow e^{-2\Delta_i}[1 + e^{-4J_i'}] \tag{A5}$$

where $J_i' = \min\{J_i, J_{i+1}\}$.

The low-temperature expansion of the last two terms in (A1) gives

$$\prod_j a_j^+ e^{-ink} + \prod_j a_j^- e^{ink} = 2e^{-2J^{ovr}} e^{-2(J_{max} - J_{min})} \cos(nk) \tag{A6}$$

where J_{max} (J_{min}) are the absolute maximum (minimum) magnitudes of the bonds in a unit cell. The parameter J^{ovr} represents the fact that as the cell is traversed the magnitudes of the bonds can increase or decrease in various regions. These regions are characterized by barriers $J_l^{max} - J_l^{min}$ and the sum over all of these is

$$\sum_{l=1}^{m} (J_l^{max} - J_l^{min}) = J^{ovr} + (J_{max} - J_{min}) \tag{A7}$$

where m is the number of local maxima (minima) in the unit cell.

The expression (A6) includes a $k = 0$ term which is cancelled by the ω-independent terms of the first term in (A1)

$$\begin{aligned} \sum_{j=0}^{j_{max}} (-1)^j A_j^n &\sim 2 \prod_j e^{-2(J_j^{max} - J_j^{min})} (1 + e^{-4J_{min}}) \\ &= 2e^{-2J^{ovr}} e^{-2(J_{max} - J_{min})} (1 + e^{-4J_{min}}). \end{aligned} \tag{A8}$$

The ω-dependent part of the secular equation for $\omega \ll 1$ is obtained using (A4). The leading small ω-dependent contribution is

$$\sum_{j=0}^{j_{max}} (-1)^j (n - 2j) A_j^n \frac{\omega}{\Gamma} \sim e^{-2J_{ovr}} \frac{\omega}{\Gamma}. \tag{A9}$$

Hence the secular equation becomes

$$e^{-2J_{ovr}} \frac{\omega}{\Gamma} \sim e^{-2J_{ovr}} e^{-2(J_{max} - J_{min})} \{1 - \cos(nk) + e^{-4J_{min}}\} \tag{A10}$$

which gives the dispersion relation (18).

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